

Orthogonality of Coastal-Trapped Waves

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ABSTRACT: Coastal-trapped wave modes are shown to be orthogonal in the sense that they make independent contributions to the energy of the wave field. The hydrostatic Boussinesq dynamics on an f plane, linearized around a state of rest, are formulated as a (generalized) Schrödinger equation, which exposes the Hermitian structure of the wave operator that implies the orthogonality of eigenmodes. This formulation, which becomes particularly simple in weak form, is parlayed into a finite-element discretization that preserves the symmetries of the original problem and therefore the energy conservation and orthogonality of modes. The geostrophic momentum approximation, under which the orthogonality of modes has been recognized previously, is reprised and discussed in the present context to emphasize that low-frequency coastal-trapped waves are edge waves. Their dynamics are governed by boundary potential vorticity anomalies, both Bretherton-type contributions familiar from the quasigeostrophic equations and lateral contributions that are important for steep slopes and coastal walls.

KEYWORDS: Coastal flows; Oceanic waves; Numerical analysis/modeling

1. Introduction

The continental margin of the ocean acts as a waveguide, supporting coastal-trapped waves that propagate with the coast to the right in the Northern Hemisphere (e.g., Huthnance et al. 1986). These trapped waves shape the coastal ocean's response to tidal and wind forcing. They can be tracked for thousands of kilometers as they propagate along continental margins after being excited by local wind events (e.g., Brink 1991), they are thought to mediate the ocean's response to tidal forcing where this forcing is subinertial (e.g., Huthnance 1974; Chapman 1989; Brink 1990; Musgrave et al. 2017), and they shape how open-ocean sea level anomalies are expressed near the coast (Hughes et al. 2019). The structure of these trapped wave modes is well characterized, and numerical approaches to their calculation have been known for decades. But a key property of these waves appears to have been overlooked: The wave modes are orthogonal in the sense that they make independent contributions to the energy of the wave field, whether or not the evolution is much slower than the inertial period. The purpose of this paper is to demonstrate this orthogonality and to formulate a simple numerical scheme that preserves this property.

The orthogonality of coastal-trapped wave modes has been recognized previously for the slow limit. I revisit these reduced dynamics based on the geostrophic momentum approximation and emphasize the edge-wave nature of the modes. In particular, I extend the notion of Bretherton's (1966) boundary potential vorticity to allow for steep slopes and coastal walls (cf. Schneider et al. 2003).

The theory of coastal-trapped waves in a continuously stratified fluid was developed by Wang and Mooers (1976), Clarke (1977), and Huthnance (1978). All these authors considered the hydrostatic Boussinesq equations on an f plane, linearized around a state of rest with a buoyancy profile that is a function

of depth only. The bathymetry has an arbitrary shape in the cross-shore direction but does not vary in the alongshore direction, such that a sinusoidal alongshore structure can be imposed ab initio. Wang and Mooers (1976) further imposed a rigid lid, whereas Clarke (1977) and Huthnance (1978) allowed for free-surface displacements. The dynamics can be posed as an eigenvalue problem, in which the eigenvalue is the frequency ω_n and the eigenfunction determines the pressure, velocity, and buoyancy fields of the mode. These authors explored the general properties of the modes, their spatial structures, and the corresponding dispersion curves for a range of idealized and realistic bathymetries and stratification profiles.

Wang and Mooers (1976) showed that these coastal-trapped wave modes are orthogonal in the limit $\omega_n^2 \ll f^2$, expressing the orthogonality both in the form of a boundary integral and as the area integral of the wave energy, with the momentum replaced by the geostrophic momentum (see also section 6). Clarke (1977) derived the same orthogonality condition in boundary integral form for waves with $\omega_n^2 \ll f^2$ and an alongshore scale that greatly exceeds the cross-shore scale, although now allowing for free-surface displacements. Huthnance (1978) restated the orthogonality condition for $\omega_n^2 \ll f^2$ and also made the connection to the wave energy, now including the potential energy due to free-surface displacements.

The question of orthogonality was raised again recently by Musgrave (2019). Seeking to understand the energetics of subinertial tides, she pointed out that coastal-trapped wave modes make independent contributions to the alongshore energy flux in the slow limit only. Kelly and Ogbuka (2022) showed that such independence can be achieved beyond the slow limit by a redefinition of the wave modes as solutions to an eigenvalue problem that treats the wave frequency as a parameter and the alongshore wavenumber as the eigenvalue (cf. Webster and Holland 1987; Johnson and Rodney 2011). But these redefined modes do not, in general, make independent contributions to the energy itself (see also section 5).

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I show below that the eigenvalue problem that considers the frequency ω_n as the eigenvalue is a (generalized) Schrödinger equation with a Hermitian operator, whether or not the slow limit is considered. The eigenvalue problem produces modes that are orthogonal in the sense that they make independent contributions to the wave energy. This form of the eigenvalue problem has the further advantage that it generalizes to more complex geometries that are not uniform in the alongshore direction. Quite generally, the response to a tidal or wind forcing can then be understood mode by mode, with no interaction between the modes in the linear equations, even in a complex geometry. For each mode, the response to the forcing depends on the projection of the forcing onto the modal structure and how close the forcing is to the resonant frequency ω_n of the mode.

The energy flux, in contrast, loses its significance in more complex geometries. For each mode, the flux is nondivergent once averaged over the period of the mode. In the presence of multiple modes, the flux emerges from the phase relation between the modes. This is how initially localized energy can spread across the domain. I therefore focus on the standard eigenvalue problem arising from the Schrödinger equation with the frequency as the eigenvalue.

2. Orthogonality of modes

Let us begin with the linearized hydrostatic Boussinesq equations in a domain adjacent to the coast at $x = 0$ and with depth $h(x)$ (Fig. 1). Although the discussion that follows is straightforward to generalize, let us consider the specific context of coastal-trapped waves by assuming an f plane and no alongshore variations in the depth. The following formulation is entirely equivalent to that employed by Wang and Mooers (1976), Clarke (1977), and Huthnance (1978), among others. Where I depart from most previous work is that I formulate the dynamics as a (generalized) Schrödinger equation, which exposes the Hermitian nature of the dynamics and the orthogonality of eigenmodes.

After a Fourier transform in the alongshore direction y and with the alongshore wavenumber denoted by l , the linearized hydrostatic Boussinesq equations read

$$\partial_t u = fv - \partial_x p, \quad (1)$$

$$\partial_t v = -fu - ilp, \quad (2)$$

$$0 = -\partial_z p + b, \quad (3)$$

$$\partial_x u + ilv + \partial_z w = 0, \quad (4)$$

$$\partial_t b = -N^2 w, \quad (5)$$

where u denotes the cross-shore velocity, v denotes the alongshore velocity, w denotes the vertical velocity, p denotes the kinematic pressure (the dynamic pressure divided by the reference density ρ_0), and b denotes the buoyancy. The stratification N^2 can vary in z . The variables u , v , w , p , and b are the complex Fourier coefficients arising from the alongshore

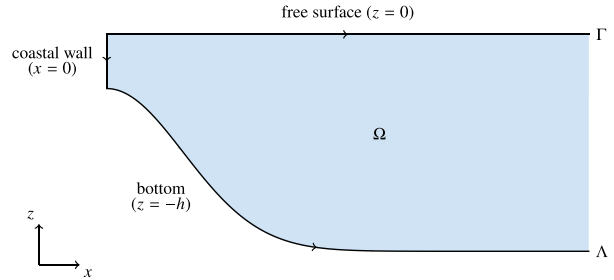


FIG. 1. Sketch of the two-dimensional domain Ω . The curve Γ marks the surface, and the curve Λ marks the coastal wall (if present) and the bottom. Both curves are oriented away from the origin.

transform. The boundary conditions are no normal flow at the bottom,

$$w = -u\partial_x h \text{ at } z = -h(x), \quad (6)$$

no flow through the coastal wall (if present), i.e., $u = 0$ at $x = 0$, and a linearized free-surface condition at the top,

$$g^{-1}\partial_t p = w \text{ at } z = 0, \quad (7)$$

where $g^{-1}p$ is the free-surface elevation. This system conserves energy:

$$\frac{dE}{dt} = 0, \quad E = \frac{1}{2} \int_{\Omega} (|u|^2 + |v|^2 + N^{-2}|b|^2) dx dz + \frac{1}{2} \int_{\Gamma} g^{-1}|p|^2 dx, \quad (8)$$

where Ω denotes the domain in (x, z) , and Γ denotes the oriented curve along $z = 0$ with orientation away from the origin (Fig. 1). The total energy consists of kinetic energy, potential energy due to interior buoyancy anomalies, and potential energy due to free-surface displacements. As stated, the energy conservation (8) holds if the energy density is integrable, which is the case if the flow variables decay sufficiently rapidly as $x \rightarrow \infty$ or if the domain is bounded at some finite x .

To better expose the structure of these linear Boussinesq equations, I recast them in operator form:

$$iM\partial_t \psi = H\psi, \quad (9)$$

with the state vector $\psi = (u, v, w, p, b)$ and the operators

$$M = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & g^{-1}\delta(z) & & \\ & & & N^{-2} & \\ & & & & \end{bmatrix}, \quad (10)$$

$$H = \begin{bmatrix} & if & & -i\partial_x & \\ -if & & & l & \\ -i\partial_x & l & -i\partial_z + i\delta(z) & -i\partial_z & i \\ & & -i & & \end{bmatrix},$$

where δ is the Dirac delta function, and only nonzero entries are shown. The equation takes the form of a Schrödinger

equation, generalized by the presence of the M operator on the left-hand side. The operator M is real, symmetric, and positive semidefinite. The operator H is Hermitian, as can be seen by forming the inner product with $\chi = (\xi, \eta, \zeta, \alpha, \beta)$ and performing integration by parts:

$$\begin{aligned} \langle \chi, H\psi \rangle &= \int_{\Omega} [if(\xi^\dagger v - \eta^\dagger u) - i(\xi^\dagger \partial_x p + \alpha^\dagger \partial_x u) + l(\eta^\dagger p + \alpha^\dagger v) \\ &\quad - i(\zeta^\dagger \partial_z p + \alpha^\dagger \partial_z w) + i(\zeta^\dagger b - \beta^\dagger w)] dx dz + \int_{\Gamma} i\alpha^\dagger w dx \\ &= \int_{\Omega} [-if(\eta^\dagger u - \xi^\dagger v) + i(\partial_x \alpha^\dagger u + \partial_x \xi^\dagger p) + l(\alpha^\dagger v + \eta^\dagger p) \\ &\quad + i(\partial_z \alpha^\dagger w + \partial_z \zeta^\dagger p) - i(\beta^\dagger w - \zeta^\dagger b)] dx dz - \int_{\Gamma} i\zeta^\dagger p dx \\ &= \langle H\chi, \psi \rangle, \end{aligned} \quad (11)$$

where † denotes complex conjugation, and I exploited the non-normal-flow boundary conditions at the bottom and the coastal wall (if present). The inner product is defined as conventional:

$$\langle \chi, \psi \rangle = \int_{\Omega} (\xi^\dagger u + \eta^\dagger v + \zeta^\dagger w + \alpha^\dagger p + \beta^\dagger b) dx dz,$$

such that $\langle \chi, a\psi \rangle = a\langle \chi, \psi \rangle$ and $\langle a\chi, \psi \rangle = a^\dagger \langle \chi, \psi \rangle$ for a complex scalar a .

Let us seek eigenmodes ψ_n of the operator pair (H, M) that satisfy

$$H\psi_n = \omega_n M\psi_n. \quad (12)$$

The eigenvalues ω_n are real because H is Hermitian and M is positive semidefinite, such that each mode oscillates with $e^{-i\omega_n t}$. Any two eigenmodes ψ_n and ψ_m with distinct frequencies $\omega_n \neq \omega_m$ are M orthogonal, as can be demonstrated by a standard argument:

$$\begin{aligned} \omega_n \langle \psi_m, \psi_n \rangle_M &= \omega_n \langle \psi_m, M\psi_n \rangle = \langle \psi_m, H\psi_n \rangle \\ &= \langle H\psi_m, \psi_n \rangle = \omega_m \langle M\psi_m, \psi_n \rangle = \omega_m \langle \psi_m, \psi_n \rangle_M, \\ \text{so } \langle \psi_m, \psi_n \rangle_M &= 0, \end{aligned} \quad (13)$$

where

$$\langle \chi, \psi \rangle_M \equiv \langle \chi, M\psi \rangle = \langle M\chi, \psi \rangle. \quad (14)$$

For degenerate eigenmodes with $\omega_n = \omega_m$ (such as geostrophic modes), an M -orthogonal basis can be found for the subspace spanned by these modes. If the motion is captured by discrete eigenmodes, i.e., if one can write

$$\psi = \sum_n a_n \psi_n e^{-i\omega_n t}, \quad (15)$$

and if the eigenmodes are normalized such that $\langle \psi_n, \psi_n \rangle_M = 1$, then the (properly orthogonalized) eigenmodes diagonalize the energy:

$$E = \frac{1}{2} \langle \psi, \psi \rangle_M = \frac{1}{2} \sum_n |a_n|^2. \quad (16)$$

Each mode makes an independent contribution to the energy. There are no cross terms. The amplitudes a_n are determined by projecting an initial condition or forcing onto the modes.

The operator pair (H, M) has the further property that it can be transformed into the pair (H', M) with a real symmetric H' by shifting the phases of u and w by $\pi/2$. This can be seen by setting $u = iu'$ and $w = iw'$ and obtaining the modified H' acting on (u', v, w', p, b) . It implies that eigenmodes can be chosen to be real in the transformed system, such that the phase must be uniform for each variable in the original system; u and w are in phase with each other and in quadrature with v, p , and b (e.g., [Huthnance 1978](#); [Kelly and Ogbuka 2022](#)). This property of coastal-trapped waves does not, however, generalize to the three-dimensional case (see [section 7](#)), and the transformation is not adopted in what follows.

Eigenmodes with distinct alongshore wavenumbers l should be considered orthogonal because of the orthogonality of their Fourier modes in the alongshore direction. While modes with distinct l are orthogonal in this three-dimensional sense, they cannot in general be expected to be M orthogonal in the two-dimensional cross-sectional sense discussed above. Consider, for example, two trapped modes with alongshore wavenumbers l and $l + \varepsilon$, denoted by ψ_n and $\psi_n^{(\varepsilon)}$. As $\varepsilon \rightarrow 0$, $\psi_n^{(\varepsilon)} \rightarrow \psi_n$ smoothly, such that $\langle \psi_n, \psi_n^{(\varepsilon)} \rangle_M \rightarrow \langle \psi_n, \psi_n \rangle_M = 1$ if the modes are normalized accordingly. Therefore, it cannot be the case that ψ_n and $\psi_n^{(\varepsilon)}$ are M orthogonal for all $\varepsilon > 0$.

It is not guaranteed that the eigenvalue problem (12) is well posed (e.g., [Sobolev 2006](#); [Maas and Lam 1995](#); [Arnold and Khesin 1998](#); [Bühler 2014](#)). The well-posedness is most easily assessed by reducing the eigenvalue problem to one equation for p_n , as is standard in the literature (cf. [section 6](#)). Assuming $\omega_n \neq 0$, the eigenvalue problem can be restated as

$$-(\omega_n^2 - f^2) \partial_z (N^{-2} \partial_z p_n) = -\partial_{xx} p_n + l^2 p_n, \quad (17)$$

with corresponding boundary conditions applied. This equation changes from elliptic for $\omega_n^2 < f^2$ to hyperbolic for $\omega_n^2 > f^2$. The elliptic subinertial case is well behaved, and modes are either geostrophic ($\omega_n = 0$) or trapped ($0 < \omega_n^2 < f^2$). The hyperbolic superinertial case can depend sensitively on the geometry. There are exceptions in semi-infinite domains, in which the superinertial modes are well defined and form a continuum of modes that propagate in from infinity, are reflected off the bathymetry, and propagate back out to infinity, but the generic case is ill posed, and I restrict myself to subinertial modes.

3. Fully prognostic formulation

The dynamics can be reduced to a purely prognostic set of equations, i.e., one that does not contain any diagnostic equations such as the hydrostatic and continuity equations of the original system. This produces a positive definite operator M and simplifies the numerical solution discussed in [section 4](#). The reduced formulation will also be the basis of the approximate slow dynamics discussed in [section 6](#). In this section, however, the dynamics remain unchanged compared to [section 2](#).

Dividing the buoyancy equation (5) by N^2 , differentiating it in z , and substituting using the hydrostatic balance (3) and the continuity equation (4) eliminate w and b :

$$\partial_{tz}(N^{-2}\partial_z p) = \partial_x u + i\ell v. \quad (18)$$

At the free surface,

$$\begin{aligned} \partial_t b = -N^2 w = -N^2 g^{-1} \partial_t p, \quad \text{so } \partial_t(N^{-2} \partial_z p + g^{-1} p) = 0 \\ \text{at } z = 0. \end{aligned} \quad (19)$$

At the bottom,

$$\partial_t b = -N^2 w = N^2 \partial_x h u, \quad \text{so } \partial_t(N^{-2} \partial_z p) = \partial_x h u \quad \text{at } z = -h. \quad (20)$$

This reduces the dynamics to the three variables u , v , and p . The eliminated variables w and b can be recovered using the continuity equation and hydrostatic balance.

One can again write the dynamics in the form

$$iM \partial_t \psi = H \psi, \quad (21)$$

with (redefined) $\psi = (u, v, p)$,

$$M = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -\partial_z(N^{-2}\partial_z \cdot) + N^{-2}\delta(z)\partial_z - N^{-2}\delta(z+h)\partial_z + g^{-1}\delta(z) \end{bmatrix}, \quad \text{and} \quad (22)$$

$$H = \begin{bmatrix} & & if & -i\partial_x \\ & -if & & l \\ -i\partial_x - i\partial_x h \delta(z+h) & & l & \end{bmatrix}. \quad (23)$$

An integration by parts in z demonstrates that $(1/2)\langle \psi, \psi \rangle_M$ still produces the correct energy. The Dirac delta function term in H is required to make it Hermitian, as can be seen by applying the Leibniz integral rule. The operator M is now positive definite—or positive semidefinite if a rigid lid is applied by omitting the $g^{-1}\delta(z)$ term. Again, the boundary terms are crucial; Kelly and Ogbuka (2022) pointed out that M without the boundary terms is not symmetric.

4. Weak formulation and finite-element approximation

Perhaps more naturally, the equations may be written in weak form (e.g., Salsa and Verzini 2022). Denoting the test functions by $\chi = (\xi, \eta, \alpha)$, the weak form of (21) is

$$i\langle \chi, \partial_t \psi \rangle_M = \langle \chi, H \psi \rangle, \quad (24)$$

or, after some integration by parts,

$$\begin{aligned} i \int_{\Omega} (\xi^\dagger \partial_t u + \eta^\dagger \partial_t v + N^{-2} \partial_z \alpha^\dagger \partial_{tz} p) dx dz + i \int_{\Gamma} g^{-1} \alpha^\dagger \partial_t p dx \\ = \int_{\Omega} [if(\xi^\dagger v - \eta^\dagger u) + i(\partial_x \alpha^\dagger u - \xi^\dagger \partial_x p) + l(\eta^\dagger p + \alpha^\dagger v)] dx dz. \end{aligned} \quad (25)$$

Conservation of energy $E = (1/2)\langle \psi, \psi \rangle_M$ is now immediate:

$$\frac{dE}{dt} = \frac{1}{2}(\langle \psi, \partial_t \psi \rangle_M^\dagger + \langle \psi, \partial_t \psi \rangle_M) = \frac{i}{2}(\langle \psi, H \psi \rangle^\dagger - \langle \psi, H \psi \rangle) = 0, \quad (26)$$

given that H is Hermitian and therefore $\langle \psi, H \psi \rangle$ is real. The weak form (25) can also be obtained directly from

a variational formulation of the problem (e.g., Salmon 2020).

The weak formulation (25) can easily be turned into a finite-element approach for numerical solutions (e.g., Strang 2007). Restricting the function space to piecewise linear functions on a triangular mesh of the domain Ω and denoting by φ_m the basis functions that are equal to 1 at the m th mesh node, 0 at all other nodes, and linear in between, one can represent the solutions in this restricted space by

$$u = \sum_m u_m \varphi_m, \quad v = \sum_m v_m \varphi_m, \quad p = \sum_m p_m \varphi_m. \quad (27)$$

Testing with the basis functions φ_n , one obtains the linear algebraic equations:

$$iB_{nm} \partial_t u_m = ifB_{nm} v_m - iG_{nm} p_m, \quad (28)$$

$$iB_{nm} \partial_t v_m = -ifB_{nm} u_m + lB_{nm} p_m, \quad (29)$$

$$i(N^{-2}K_{nm} + g^{-1}S_{nm}) \partial_t p_m = iG_{nm} u_m + lB_{nm} v_m, \quad (30)$$

where

$$B_{nm} = \int_{\Omega} \varphi_n \varphi_m dx dz, \quad G_{nm} = \int_{\Omega} \partial_x \varphi_n \varphi_m dx dz, \quad (31)$$

$$K_{nm} = \int_{\Omega} \partial_z \varphi_n \partial_z \varphi_m dx dz, \quad S_{nm} = \int_{\Gamma} \varphi_n \varphi_m dx. \quad (32)$$

One is then left with the finite-dimensional eigenvalue problem

$$\mathbf{H} \psi_n = \omega_n \mathbf{M} \psi_n, \quad (33)$$

with a finite-dimensional eigenvector ψ_n and the matrices

$$\mathbf{M} = \begin{bmatrix} \mathbf{B} & & \\ & \mathbf{B} & \\ & & N^{-2}\mathbf{K} + g^{-1}\mathbf{S} \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} & if\mathbf{B} & -i\mathbf{G}^T \\ & -if\mathbf{B} & l\mathbf{B} \\ i\mathbf{G} & l\mathbf{B} & \end{bmatrix}. \quad (34)$$

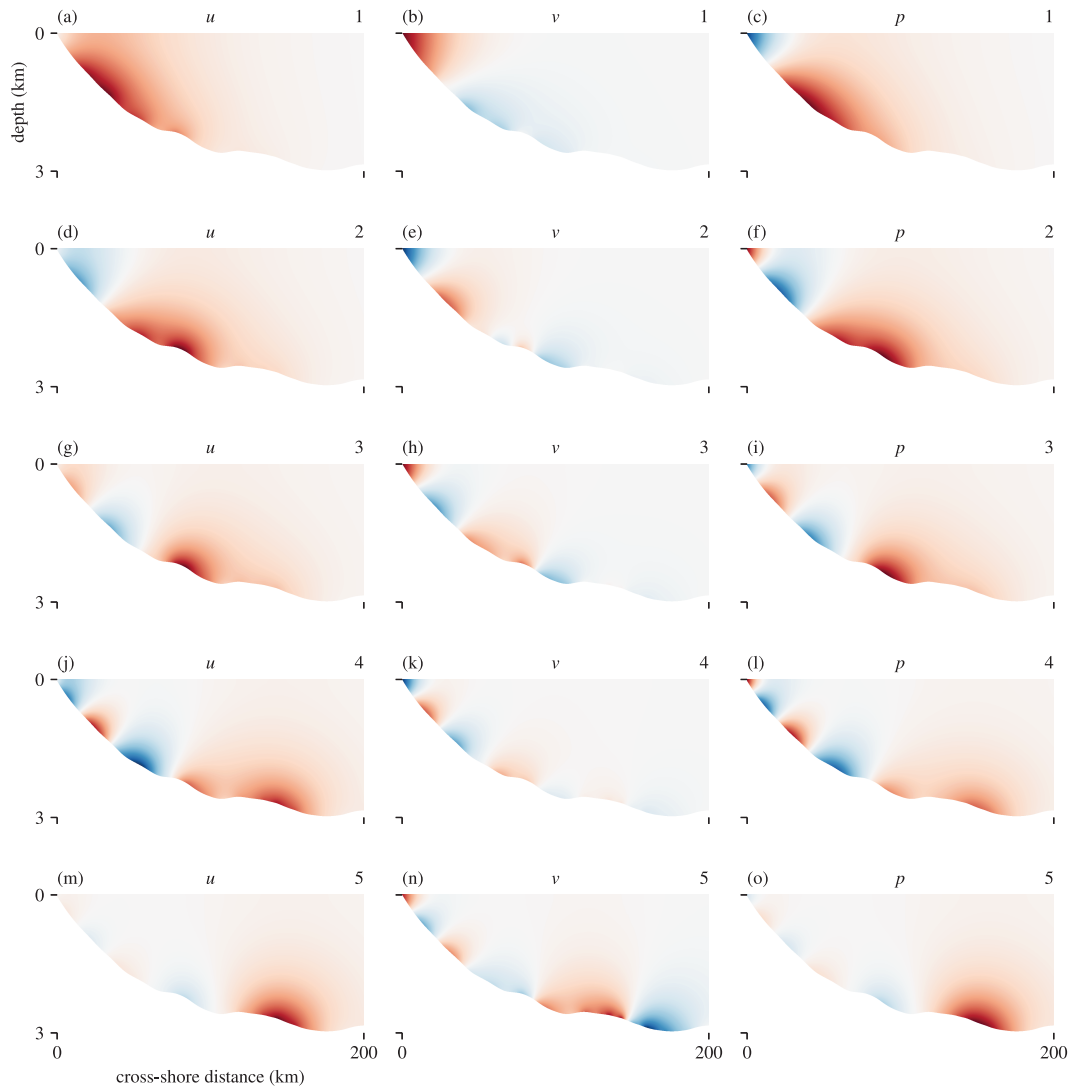


FIG. 2. Coastal-trapped wave modes 1–5 for a randomly perturbed exponential bathymetry and uniform stratification. These orthogonal modes for an alongshore wavenumber $l = 2\pi/200$ km have frequencies $\omega_n/f = -0.52, -0.31, -0.25, -0.19$, and -0.18 . Only the real part of the cross-shore velocity u and the imaginary parts of the alongshore velocity v and pressure p are shown because the respective other parts vanish. The color scale is normalized to the maximum in each panel. Only the first half of the 400-km-wide domain is displayed.

The matrix $\mathbf{M}$ is symmetric and positive definite, and the matrix $\mathbf{H}$ is Hermitian.

This finite-element discretization thus preserves the symmetry of the operators M and H , such that eigenvalues are still guaranteed to be real and the modes are exactly orthogonal. Results from the discrete system should still be interpreted carefully, however, given the discussion of well-posedness above. In particular, superinertial eigenvalues and the corresponding eigenmodes cannot in general be expected to converge as the mesh is refined. Subinertial modes, in contrast, behave well and exhibit the expected convergence (appendix). No spurious subinertial modes arising from the discretization have been observed, with and without a rigid lid.

5. Examples

To illustrate the approach, exhibit typical mode structures, and confirm agreement with previous calculations, I provide a few examples of subinertial coastal-trapped wave modes using the finite-element method described in the previous section. I calculate modes for a bumpy bathymetry, a linear bathymetry following Huthnance (1978), and a tanh bathymetry following Kelly and Ogbuka (2022).

First, let us consider a randomly perturbed exponential bathymetry $h(x) = h_0(1 - e^{-x/d})$ with a decay scale $d = 50$ km (Fig. 2). The (additive) random depth perturbations are drawn from a Gaussian PDF using a Matérn covariance function (e.g., Stein 1999) with correlation distance $\rho = 25$ km,

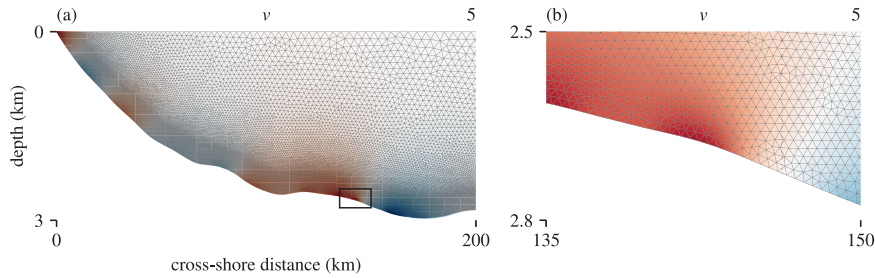


FIG. 3. Triangular mesh used in the finite-element calculation of orthogonal coastal-trapped wave modes. (left) Half of the domain and (right) a zoom-in to the small fraction of the domain indicated by the rectangle in the left panel. The shading shows the alongshore velocity field for mode 5 (see Fig. 2n).

smoothness parameter $\nu = 5/2$, and standard deviation $\sigma = \min\{h(x)/10, 150 \text{ m}\}$. I use a uniform buoyancy frequency $N = 3 \times 10^{-3} \text{ s}^{-1}$ for simplicity, an inertial frequency $f = 10^{-4} \text{ s}^{-1}$, and an offshore depth of $h_0 = 3 \text{ km}$.

I generate a mesh in nondimensional space with z normalized by h_0 and x normalized by $\lambda = Nh_0/f = 90 \text{ km}$, which produces a mesh aspect ratio near the Prandtl ratio f/N (Fig. 3). I refine the mesh near the bottom, where shorter scales must be resolved. I use a nondimensional characteristic mesh size of 0.05 times the distance from the bottom but limit it to a minimum of 0.005, corresponding to 15 m in the vertical and 450 m in the horizontal. I use a dimensional domain width of 400 km to allow modes to decay sufficiently offshore, such that they are not affected appreciably by an artificial offshore wall. This produces a mesh with 41 907 nodes (Table 1).

An implicitly restarted Arnoldi iteration applied to $(\mathbf{H} - \omega_0 \mathbf{M})^{-1} \mathbf{M}$ (e.g., Bai et al. 2000) performs well for the subinertial modes. With a shift ω_0 not too far from the target eigenvalues ($\omega_0 = -f/2$ works well for the modes shown in Fig. 2), the Arnoldi iterations for the first five eigenvalues converge in a handful of iterations and take less than 10 s on a laptop.

The structure of the modes (Fig. 2) is consistent with expectations and previous calculations (e.g., Wang and Mooers 1976; Huthnance 1978; Musgrave 2019). The modes for an alongshore wavenumber $l = 2\pi/200 \text{ km}$ are bottom trapped, and the cross-shore structure decreases in scale with mode number. The number of bottom zero crossings in the pressure modes equals the mode number.

Second, I follow Huthnance (1978) in choosing a linear bathymetric slope that transitions to a flat bottom offshore (Fig. 4). I use the same N , f , and h_0 as before. The slope h_0/d is chosen such that the slope Burger number is $\lambda^2/d^2 = 1/2$, i.e., the transition to a flat bottom occurs at $x = d = \sqrt{2}\lambda = 127 \text{ km}$. For the reference case, I use an alongshore wavenumber $l = \lambda^{-1}$.

TABLE 1. Mesh parameters for the three types of bathymetry considered in this paper. The linear bathymetry is taken from Huthnance (1978), and the tanh bathymetry is taken from Kelly and Ogbuka (2022).

Bathymetry	Domain width (km)	Nodes	Elements
Bumpy	400	41 907	82 629
Linear	4000	15 958	31 228
Tanh	400	43 205	85 157

I use a 4000-km-wide domain to allow for sufficient offshore decay even for small l . I make the mesh size the same function of the distance from the sloping bottom as before, which produces a very coarse mesh far offshore, where the wave modes vary slowly. The number of mesh nodes is smaller than in the previous example because the sloping bathymetry over which the mesh is refined is now confined to $x < 127 \text{ km}$ (Table 1).

The modes again show the expected bottom-trapped structure with an increase in the number of nodes with the mode number (Fig. 4). I also vary the alongshore wavenumber to obtain the dispersion curves for the first five modes (Fig. 5), which are consistent with Huthnance (1978). Again, the presently calculated modes do not, in fact, differ from those previously obtained, aside from differences in the numerical approach. Previously, the modes were simply not recognized as orthogonal in the sense discussed here.

The present finite-element discretization can also be used to solve the eigenvalue problem that treats ω as a parameter and the alongshore wavenumber l_n as the eigenvalue (e.g., Webster and Holland 1987; Johnson and Rodney 2011; Kelly and Ogbuka 2022). In weak form, this eigenvalue problem reads

$$\langle \chi, \tilde{H} \tilde{\psi}_n \rangle = l_n \langle \chi, \tilde{\psi}_n \rangle_{\tilde{M}}, \quad (35)$$

where the operators are defined by

$$\begin{aligned} \langle \chi, \tilde{H} \psi \rangle &= \omega \int_{\Omega} (\xi^{\dagger} u + \eta^{\dagger} v + N^{-2} \partial_z \alpha^{\dagger} \partial_z p) dx dz \\ &+ \omega \int_{\Gamma} g^{-1} \alpha^{\dagger} p dx - \int_{\Omega} [if(\xi^{\dagger} v - \eta^{\dagger} u) \\ &+ i(\partial_x \alpha^{\dagger} u - \xi^{\dagger} \partial_x p)] dx dz, \quad \text{and} \end{aligned} \quad (36)$$

$$\langle \chi, \psi \rangle_{\tilde{M}} \equiv \langle \chi, \tilde{M} \psi \rangle = \int_{\Omega} (\eta^{\dagger} p + \alpha^{\dagger} v) dx dz. \quad (37)$$

The discrete eigenvalue problem is then

$$\tilde{H} \tilde{\psi}_n = l_n \tilde{M} \tilde{\psi}_n, \quad \tilde{M} = \begin{bmatrix} & & \\ & \mathbf{B} & \\ \mathbf{B} & & \end{bmatrix}, \quad \tilde{H} = \begin{bmatrix} \omega \mathbf{B} & -if \mathbf{B} & i \mathbf{G}^T \\ if \mathbf{B} & \omega \mathbf{B} & \\ -i \mathbf{G} & & \omega(N^{-2} \mathbf{K} + g^{-1} \mathbf{S}) \end{bmatrix}. \quad (38)$$

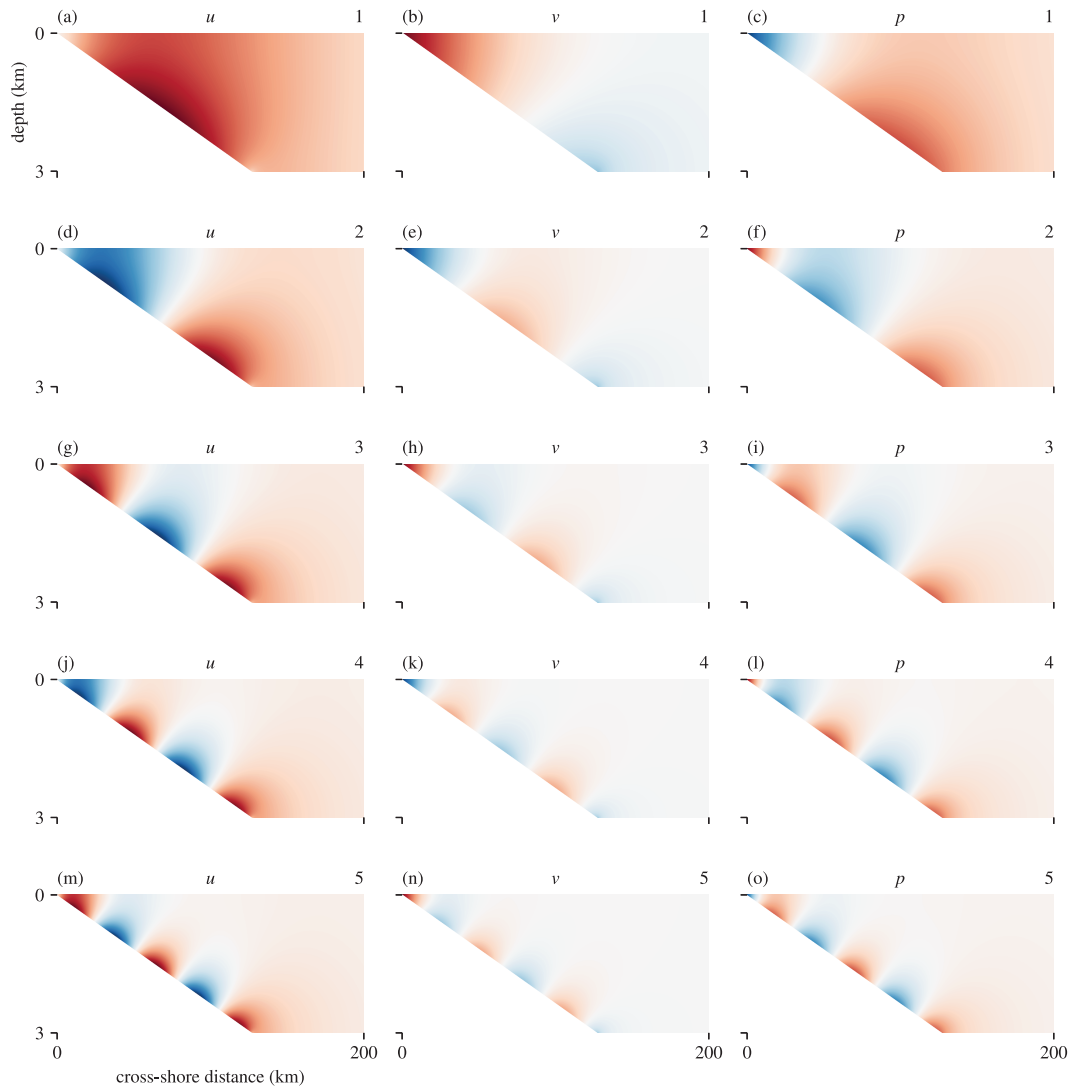


FIG. 4. Coastal-trapped wave modes 1–5 for a bathymetry with a constant slope, uniform stratification, a rigid lid, and a slope Burger number of $1/2$ (following Huthnance 1978). These orthogonal modes for an alongshore wavenumber $l = \lambda^{-1}$ have frequencies $\omega_n/f = -0.35, -0.18, -0.12, -0.085$, and -0.067 . Only the real part of the cross-shore velocity u and the imaginary parts of the alongshore velocity v and pressure p are shown because the respective other parts vanish. The color scale is normalized to the maximum in each panel. Only the first 200 km of the 4000-km-wide domain is displayed.

I discretize the idealized tanh geometry described by Kelly and Ogbuka (2022) as before, except that I now also refine the mesh near the coastal wall. I prescribe $\omega = 2\pi/24$ h and solve using a shift-and-invert approach with a shift $l_0 = 2\pi/200$ km (Fig. 6). I obtain alongshore wavelengths that are close to those reported by Kelly and Ogbuka (2022), $2\pi/l_1 = 326$ km and $2\pi/l_2 = 128$ km, and I have ensured that these values are converged. As noted previously, these two modes with distinct l_n are not M orthogonal. They are $\tilde{M}$ orthogonal and make independent contributions to the alongshore energy flux $(1/2)\langle\tilde{\psi}, \tilde{\psi}\rangle_{\tilde{M}}$ (Kelly and Ogbuka 2022).

6. Slow motion

That coastal-trapped wave modes are orthogonal has previously been recognized for the $\omega_n^2 \ll f^2$ limit. I briefly recapitulate this result, treating it in the same framework as used in the rest of the paper, and I apply the same finite-element discretization to it. The discussion emphasizes that subinertial coastal-trapped waves are best thought of as bottom edge waves.

Let us begin by making the geostrophic momentum approximation (cf. Hoskins and Bretherton 1972; Hoskins 1975):

$$-ilf^{-1}\partial_y p = f\mathbf{v} - \partial_x p, \quad (39)$$

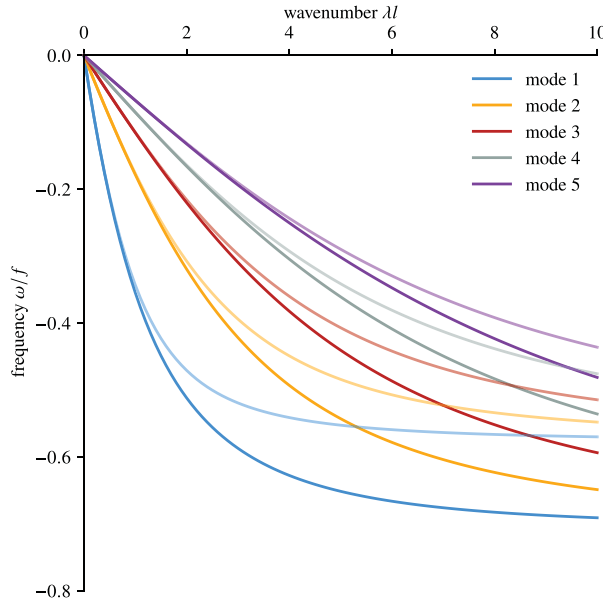


FIG. 5. Dispersion curves for coastal-trapped modes 1–5 for a bathymetry with a constant slope, uniform stratification, a rigid lid, and a slope Burger number of $1/2$. The solid lines show the eigenvalues of the full problem, and the transparent lines show the eigenvalues of the approximate slow problem (section 6). The lines for the first two modes can be compared to the corresponding ones in Fig. 8 of Huthnance (1978), although there l is scaled by d instead of λ and a smaller wavenumber range is displayed.

$$f^{-1}\partial_{xx}p = -fu - ilp, \quad (40)$$

$$\partial_{tz}(N^{-2}\partial_z p) = \partial_x u + ilv. \quad (41)$$

Cross differentiating the momentum equations yields the vorticity equation:

$$f^{-2}\partial_t(-\partial_{xx} + l^2)p = \partial_x u + ilv, \quad (42)$$

and combining it with (41) produces

$$\partial_t[f^{-2}(-\partial_{xx} + l^2)p - \partial_z(N^{-2}\partial_z p)] = 0. \quad (43)$$

This is a conservation statement for interior geostrophic potential vorticity, linearized around a state of rest. If the background stratification is nonuniform, the potential vorticity should be understood as a linearization of geostrophic available potential vorticity (Wagner and Young 2015). In the absence of interior potential vorticity gradients, as on the f plane and with no mean flow, as considered here, the dynamics are controlled by potential vorticity anomalies at the boundary (cf. Schneider et al. 2003). At the bottom, adding the buoyancy equation (20) and $f^{-1}\partial_x h$ times (40) yields

$$\partial_t(f^{-2}\partial_x h\partial_x p + N^{-2}\partial_z p) = -ilf^{-1}\partial_x hp \text{ at } z = -h. \quad (44)$$

At the surface, as before,

$$\partial_t(N^{-2}\partial_z p + g^{-1}p) = 0 \text{ at } z = 0. \quad (45)$$

Equations (44) and (45) can be read as conservation statements for boundary potential vorticity and combined with the interior potential vorticity equation (43):

$$\partial_t Mp = Hp, \quad (46)$$

with

$$M = f^{-2}(-\partial_{xx} + l^2) - \partial_z(N^{-2}\partial_z \cdot) + N^{-2}\delta(z)\partial_z - \delta(z+h)(f^{-2}\partial_x h\partial_x + N^{-2}\partial_z) + g^{-1}\delta(z), \text{ and} \quad (47)$$

$$H = -lf^{-1}\partial_x h\delta(z+h). \quad (48)$$

The operators M and H now act on p , the only remaining prognostic variable. With the usual convention, $-fMp$ is the potential vorticity. The interior potential vorticity has no

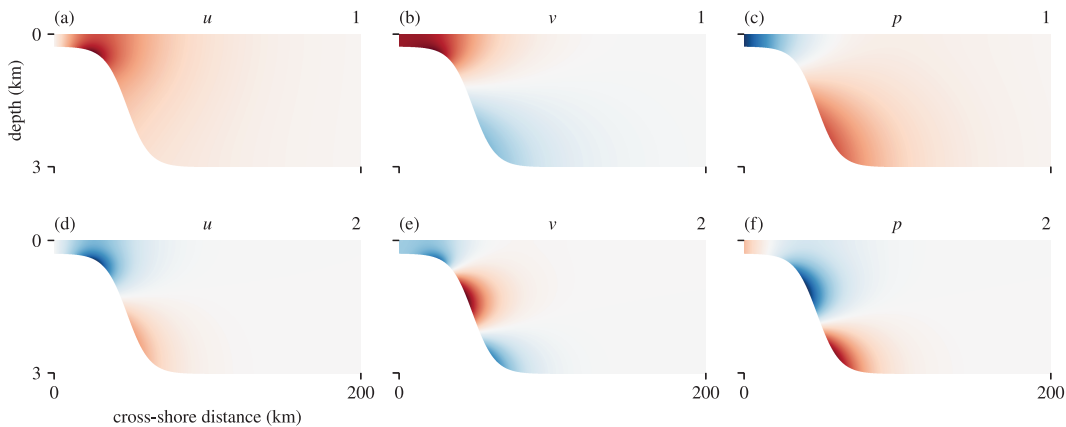


FIG. 6. Eigenmodes of (38) for the idealized shelf geometry of Kelly and Ogbuka (2022) and a diurnal tidal period of 24 h. These modes have alongshore wavelengths of 326 and 128 km. Only the real part of the cross-shore velocity u and the imaginary parts of the alongshore velocity v and pressure p are shown because the respective other parts vanish. Only the first half of the 400-km-wide domain is displayed.

sources or sinks. The bottom boundary potential vorticity consists of a [Bretherton \(1966\)](#) sheet $fN^{-2}\partial_x p$ and a lateral piece $f^{-1}\partial_x h\partial_x p$, which originates analogously from the relative vorticity rather than the stretching term. This boundary potential vorticity is generated by the bottom geostrophic flow $-ilf^{-1}p$ across the topographic slope $\partial_x h$. The surface boundary potential vorticity consists of a [Bretherton \(1966\)](#) sheet $-fN^{-2}\partial_x p$ and the sheet $-fg^{-1}p$ that represent the stretching due to free-surface displacements. There are no sources or sinks of surface potential vorticity in this setup. All wave modes are bottom edge waves (cf. [Schmidt and Johnson 1993](#)).

If a coastal wall is present, the flow has to additionally satisfy

$$u = 0, \text{ so } f^{-1}\partial_x p = -ilp \text{ at } x = 0. \quad (49)$$

In this case,

$$M = f^{-2}(-\partial_{xx} + l^2) - \partial_z(N^{-2}\partial_z \cdot) - f^{-2}\delta(x)\partial_x + N^{-2}\delta(z)\partial_z - \delta(z+h)(f^{-2}\partial_x h\partial_x + N^{-2}\partial_z) + g^{-1}\delta(z), \text{ and} \quad (50)$$

$$H = -lf^{-1}\delta(x) - lf^{-1}\partial_x h\delta(z+h). \quad (51)$$

The potential vorticity $-fMp$ now includes the additional term $f^{-1}\partial_x p\delta(x)$ representing the contribution from the coastal wall, and there is an additional source term produced by the geostrophic flow $-ilf^{-1}p$ across the wall. (The boundary condition at $x = 0$ is $u = 0$, not $-ilf^{-1}p = 0$.) All boundary potential vorticity contributions, except for that due to the free-surface displacement, originate from the boundary-normal component of the vector field $\mathbf{D} = f^{-2}\partial_x p\mathbf{x} + N^{-2}\partial_z p\mathbf{z}$ ([Clarke 1977](#); [Schneider et al. 2003](#)), although neither of the two gauges Schneider et al. discussed produces this $\mathbf{D}$ upon approximation. The boundary contributions are needed to absorb the boundary terms that arise from the integration by parts of the interior potential vorticity when forming the energy equation (see below).

The lateral contributions to the boundary potential vorticity, $f^{-1}\partial_x h\partial_x p\delta(z+h)$ for a sloping bottom and $f^{-1}\partial_x p\delta(x)$ for a coastal wall, are absent in a quasigeostrophic analysis of edge waves ([Rhines 1970](#)). This is because the quasigeostrophic scaling requires topographic slopes to be much smaller than f/N . The neglect of these lateral terms means that lateral edge waves such as Kelvin waves are not captured. In contrast, the present formulation based on the geostrophic momentum approximation captures the potential vorticity sheet at the coastal wall and produces the following Kelvin wave dispersion relation (imposing a flat bottom at $z = -h_0$):

$$\frac{\omega_n}{f} = -\frac{l}{\sqrt{l^2 + \lambda_n^{-2}}}, \quad (52)$$

where λ_n is the n th deformation radius (cf. [Kushner et al. 1998](#)). For uniform stratification and with a rigid lid, for example, $\lambda_n = \lambda/n\pi$ with $\lambda = Nh_0/f$. For long waves ($l^2 \ll \lambda_n^{-2}$), the condition $\omega_n^2 \ll f^2$ is satisfied, and the approximate dispersion

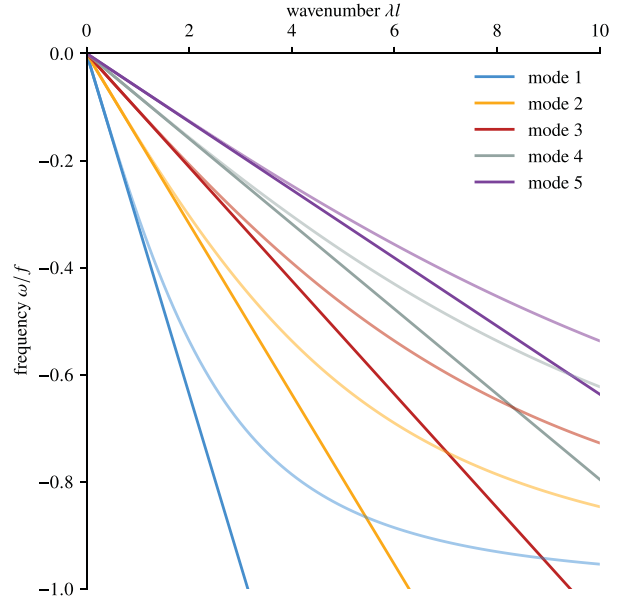


FIG. 7. Internal Kelvin wave dispersion curves for the full system (solid lines) and the slow dynamics based on the geostrophic momentum approximation (transparent lines). Note that the lines converge for $\lambda l \rightarrow 0$.

relation (52) approaches the Kelvin wave dispersion relation $\omega_n/f = -\lambda_n l$ of the full system (Fig. 7).

Whether or not a coastal wall is present, the geostrophic momentum approximation preserves the Hermitian structure of the original system. The system (46) conserves the energy

$$E = \frac{1}{2} \langle p, p \rangle_M = \frac{1}{2} \int_{\Omega} [f^{-2}(|\partial_x p|^2 + l^2|p|^2) + N^{-2}|\partial_z p|^2] dx dz + \frac{1}{2} \int_{\Gamma} g^{-1}|p|^2 dx. \quad (53)$$

In this expression, geostrophic momentum replaces the momentum that appears in the full energy, as implied by the initial approximation. The approximate system's eigenmodes are M orthogonal, which is to be understood with respect to the redefined M . The eigenmodes make independent contributions to the approximated energy (53).

One can again formulate the dynamics in weak form, either by testing the approximated strong form (46) or by making the geostrophic momentum approximation in the weak form (25) of the full dynamics:

$$i \int_{\Omega} [f^{-2}(\partial_x \alpha^\dagger \partial_x p + l^2 \alpha^\dagger p) + N^{-2} \partial_z \alpha^\dagger \partial_z p] dx dz + i \int_{\Gamma} g^{-1} \alpha^\dagger \partial_x p dx = \int_{\Lambda} lf^{-1} \alpha^\dagger p dz, \quad (54)$$

where Λ is the oriented curve that consists of the coastal wall (if present) and the bottom, and its orientation is pointing away from the origin (Fig. 1). All boundary conditions are enforced naturally, substantially simplifying the weak form compared to the strong form. Compared to the weak form of the

full problem, (54) is also simplified substantially because the three terms in the integrand on the right-hand side of (25) combine into a single boundary integral. This reduction to an integral along the coastal wall and bottom again indicates that all waves are edge waves in the slow dynamics. The geostrophic momentum approximation has removed inertia-gravity waves.

The weak form (54) allows writing the orthogonality condition in an alternative form. If $\omega_m \neq 0$, then

$$\langle p_n, p_m \rangle_M = \langle p_n, M p_m \rangle = \omega_m^{-1} \langle p_n, H p_m \rangle, \quad (55)$$

and the orthogonality condition can be expressed as a boundary integral:

$$\int_{\Lambda} p_n^\dagger p_m dz = 0, \quad (56)$$

if $\omega_n \neq \omega_m$ and $l \neq 0$ (e.g., Wang and Mooers 1976; Clarke 1977; Huthnance 1978; Brink 1989). In this commonly quoted form of the orthogonality condition, however, it is not immediately apparent that the modes make independent contributions to the energy. It is the latter property that carries over to the full system in which the geostrophic momentum approximation is not made.

In the approximate slow dynamics, the eigenmodes have the additional property that they make independent contributions not only to the energy $E = (1/2) \langle p, p \rangle_M$ but also to the along-shore energy flux. Setting $\tilde{M} = l^{-1}H$, the flux can be written as

$$\begin{aligned} F &= \frac{1}{2} \langle p, p \rangle_{\tilde{M}} = \frac{1}{2} \int_{\Lambda} f^{-1} |p|^2 dz = \frac{1}{2} \int_{\Omega} f^{-1} \partial_x p |p|^2 dx dz \\ &= \frac{1}{2} \int_{\Omega} f^{-1} (p^\dagger \partial_x p + \partial_x p^\dagger p) dx dz, \end{aligned} \quad (57)$$

showing that this indeed recovers the pressure work by the geostrophic alongshore flow. The modes make independent contributions to F because

$$\begin{aligned} \langle p_n, p_m \rangle_{\tilde{M}} &= \langle p_n, \tilde{M} p_m \rangle = \langle p_n, l^{-1} H p_m \rangle = l^{-1} \langle p_n, \omega_m M p_m \rangle \\ &= \frac{\omega_m}{l} \langle p_n, p_m \rangle_M, \end{aligned} \quad (58)$$

such that the M orthogonality of the modes implies $\tilde{M}$ orthogonality. As discussed by Musgrave (2019), who pointed out the simultaneous orthogonality, this result entails advantages in the interpretation of alongshore energy fluxes that do not carry over to the full dynamics. The discussion here suggests that the simultaneous orthogonality is rather fortuitous and cannot be expected to apply to the full system. As mentioned in the introduction, this may not be too important, however, given that the energy flux loses its importance in more general geometries.

As before, one immediately obtains a finite-element approximation of the eigenvalue problem in weak form:

$$\begin{aligned} \mathbf{H} \mathbf{p}_n &= \omega_n \mathbf{M} \mathbf{p}_n, \quad \mathbf{M} = f^{-2} (\mathbf{L} + l^2 \mathbf{B}) + N^{-2} \mathbf{K} + g^{-1} \mathbf{S}, \\ \mathbf{H} &= l f^{-1} \mathbf{P}, \end{aligned} \quad (59)$$

where now the eigenvector $\mathbf{p}_n$ contains the nodal values of the pressure field only. The additional matrices are defined as

$$L_{nm} = \int_{\Omega} \partial_x \varphi_n \partial_x \varphi_m dx dz, \quad P_{nm} = \int_{\Lambda} \varphi_n \varphi_m dz. \quad (60)$$

Both of these matrices are symmetric, and $\mathbf{L}$ is positive semi-definite. The eigenvalue problem (59) is somewhat easier to solve than the full problem (33), given the factor of 3 reduction in matrix size. The resulting eigenvalues are good approximations to the frequencies of the full modes if $\omega_n^2 \ll f^2$ (Fig. 5). For the example with a bumpy exponential bathymetry (Fig. 2), the approximate eigenmodes at $l = 2\pi/200$ km ($\lambda l = 2.8$) take a very similar shape as the full modes, even for the first two modes for which ω_n^2/f^2 is not particularly small (Fig. 8).

7. Discussion

It is straightforward to generalize the dynamics to three dimensions. The eigenvalue problem of the hydrostatic Boussinesq dynamics, written in fully prognostic form, is

$$\langle \chi, H \psi_n \rangle = \omega_n \langle \chi, \psi_n \rangle_M. \quad (61)$$

This is again written in weak form and must hold for any test function χ . The symmetric positive definite M and the Hermitian H are defined by

$$\begin{aligned} \langle \chi, \psi \rangle_M &= \int_{\Omega} (\xi^\dagger u + \eta^\dagger v + N^{-2} \partial_z \alpha^\dagger \partial_z p) dx dy dz \\ &+ \int_{\Gamma} g^{-1} \alpha^\dagger p dx dy, \quad \text{and} \end{aligned} \quad (62)$$

$$\begin{aligned} \langle \chi, H \psi \rangle &= i \int_{\Omega} [f (\xi^\dagger v - \eta^\dagger u) + \partial_x \alpha^\dagger u - \xi^\dagger \partial_x p + \partial_y \alpha^\dagger v \\ &- \eta^\dagger \partial_y p] dx dy dz, \end{aligned} \quad (63)$$

where Ω now denotes the three-dimensional domain and Γ denotes the two-dimensional surface at $z = 0$. The strong form of these operators arises in complete analogy to the two-dimensional case discussed in section 3, now treating both the x and y directions like the x direction there. The eigenvalue problem continues to be well posed for $\omega_n^2 < f^2$ and yields M -orthogonal modes. The weak form (61) is again amenable to finite-element discretization, and a piecewise linear basis on a tetrahedral mesh is expected to behave like the two-dimensional version discussed above. Variations in N and f can be taken into account by absorbing them into the definition of the associated finite-element matrix.

The slow dynamics can also be generalized to three dimensions. The eigenvalue problem is

$$\langle \alpha, H p_n \rangle = \omega_n \langle \alpha, p_n \rangle_M, \quad (64)$$

for all test functions α , where

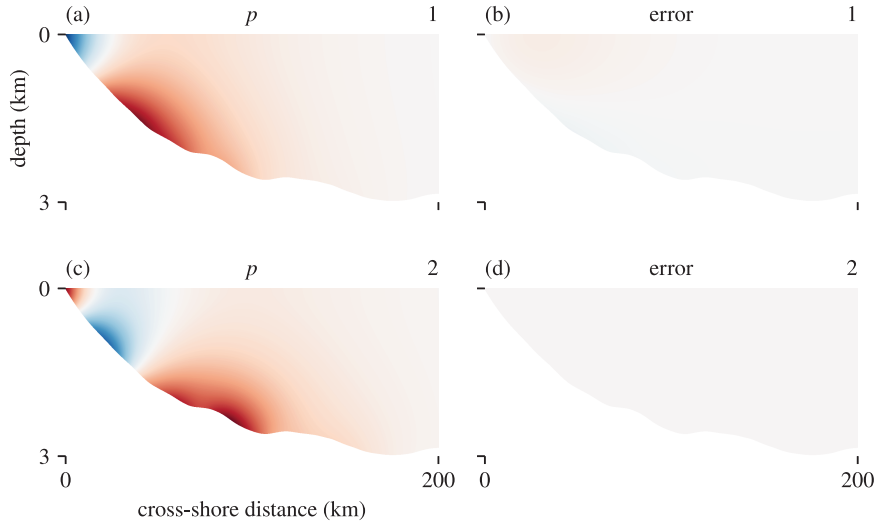


FIG. 8. Approximate modes based on the geostrophic momentum approximation for the setup of Fig. 2. (left) The first two approximate modes obtained using (59), and (right) the differences with the full modes. The numerical error is much smaller than the error due to the approximate dynamics. The color scale is adjusted to the maximum of each mode, and the same color scale is used for the error. The frequencies of the approximate modes are $\omega_n/f = -0.46$ and -0.30 , not far from the $\omega_n/f = -0.52$ and -0.31 of the full modes.

$$\langle \alpha, p \rangle_M = \int_{\Omega} [f^{-2}(\partial_x \alpha^\dagger \partial_x p + \partial_y \alpha^\dagger \partial_y p) + N^{-2} \partial_z \alpha^\dagger \partial_z p] dx dy dz + \int_{\Gamma} g^{-1} \alpha^\dagger p dx dy, \quad \text{and} \quad (65)$$

$$\langle \alpha, Hp \rangle = i \int_{\Lambda} f^{-1} \alpha^\dagger \frac{\partial(p, h)}{\partial(x, y)} dx dy, \quad (66)$$

assuming for simplicity that there are no coastal walls and that we remain on an f plane. As in two dimensions, the cross-isobath geostrophic flow provides the only source or sink of potential vorticity.

To extend the modal approach pursued here to superinertial frequencies, where the modal problem as formulated above is ill posed in general, one would have to add viscous and diffusion terms to the equations of motion. This would cause energy dissipation and thus spoil the Hermitian nature of the operator. Eigenmodes would not be orthogonal anymore. Calculating left and right eigenvectors and taking advantage of their biorthogonality might alleviate this complication and allow a modal decomposition of arbitrary motion. This framework might be used gainfully to understand the ocean's response to tidal forcing, a response in which mode interactions are restricted to nonlinear effects.

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Data availability statement. The source code for the finite-element calculations and mesh generation can be found at <https://github.com/joernc/ctw>.

APPENDIX

Convergence

I assess the convergence of the finite-element scheme by considering internal Kelvin waves in uniform stratification and with a free surface. With a flat bottom and in a semi-infinite domain, the frequencies of the Kelvin wave modes are $\omega_n/f = -\lambda_n l$. The deformation radii λ_n are determined by finding the roots of

$$\frac{N h_0}{f \lambda_n} \tan \frac{N h_0}{f \lambda_n} = \frac{N^2 h_0}{g}, \quad (A1)$$

which are approximately

$$\lambda_n \simeq \frac{N h_0}{n \pi f} \left(1 - \frac{N^2 h_0}{n^2 \pi^2 g} \right) \quad \text{for } n = 1, 2, \dots \text{ if } \frac{N^2 h_0}{g} \ll 1. \quad (A2)$$

I solve the discrete eigenvalue problem (33) using meshes of a 1000-km-wide domain with characteristic mesh sizes varying from 0.005 to 0.08 in nondimensional units. The parameters are set to the same values as in the main text: $N = 3 \times 10^{-3} \text{ s}^{-1}$, $h_0 = 3 \text{ km}$, $f = 10^{-4} \text{ s}^{-1}$, and $g = 10 \text{ m s}^{-2}$.

The eigenvalues converge, as expected, with the square of the characteristic mesh size (Fig. A1). Low modes have a smaller relative error because they vary on a larger scale than high modes. The same convergence rate applies to the

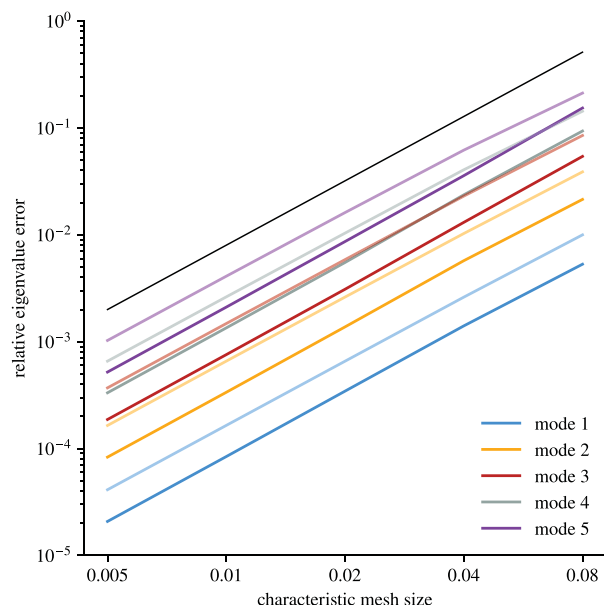


FIG. A1. Convergence analysis for the linear finite-element discretization using internal Kelvin wave modes. Shown are the errors of the numerical eigenvalues relative to the exact eigenvalues ω_n/f for the first five modes across different characteristic mesh sizes. The nondimensional mesh sizes are indicated on the horizontal axis. The solid lines show the results for the full dynamics, and the transparent lines show the results for the approximate slow dynamics. The black reference line shows a mesh size squared scaling.

approximate slow problem (59), as assessed by comparing the numerical eigenvalues to the exact eigenvalues (52) (Fig. A1). If a more rapid convergence is desired, a higher-order finite-element scheme can be used.

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